Distance Pebbling on Directed Cycle Graphs

Michael P. Knapp

1 Introduction

Suppose that G is a graph, and imagine that we have placed pebbles at some of the vertices of G. Traditionally, a **pebbling move** on G is defined as follows. If a vertex v of G contains at least two pebbles, then we may remove two pebbles from v, and add one pebble to a vertex adjacent to v. (If we have a directed graph, then we may only add a pebble to a vertex w if there is an edge which goes from v to w.) The **pebbling number** of G is then defined as the smallest number s with the property that if the initial configuration contains at least s pebbles, then no matter how they are configured it is always possible to find a sequence (possibly empty) of pebbling moves which places a pebble on any specified vertex. There are many variations of the pebbling number which can be studied, and [3] is an excellent reference.

For our purposes, we make a cosmetic change in the definition of a pebbling move. Instead of removing two pebbles from v and then adding an entirely new pebble to the graph, we will remove one pebble from v and then *move* another pebble from v to an adjacent vertex. While the two definitions are obviously equivalent in terms of which configurations of pebbles can arise on the graph, our definition allows us to follow a particular pebble through a sequence of pebbling moves. For a positive integer d, we define the **distance d pebbling number** of G to be the smallest number s such that if the initial configuration contains at least s pebbles, then there must exist a pebble which can be moved to a vertex which is at a distance of at least d from its

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starting point. This can also be thought of as finding the **target pebbling number** $\pi^{-}(G, \mathscr{D})$ (see the introduction to [3]), where $\mathscr{D}$ is a particular set of configurations of pebbles on G.

In this article, we will determine the distance d pebbling numbers for a directed cycle graph with n vertices. Let us denote this number by $P_n(d)$. Then we have the following theorem.

Theorem 1. *Suppose that $d < n$, and write $n = dq + r$, where q, r are integers with $0 \leq r \leq d-1$. Then we have $P_n(d) = \left(2^d - 1\right) q + 2^r$.*

It is not too hard to see that $P_n(d)$ must be at least as large as the bound in the theorem. Label the vertices of the graph consecutively as $v_0, \ldots, v_{n-1}$. Put $2^d - 1$ pebbles on each of $v_0, v_d, v_{2d}, \ldots, v_{(q-1)d}$, and put $2^r - 1$ pebbles on v_{qd}. Then it is not hard to see that none of the pebbles on v_0 can be moved to v_d, none of the pebbles on v_d can be moved to v_{2d}, and so on. Moreover, none of the pebbles on v_{qd} can be moved to v_0. Our initial configuration involves $\left(2^d - 1\right) q + 2^r - 1$ pebbles, but no pebble can be moved d vertices from its starting point. Hence we must have $P_n(d) \geq \left(2^d - 1\right) q + 2^r$. To complete the proof of the theorem, it suffices to show that in any initial configuration of exactly $\left(2^d - 1\right) q + 2^r$ pebbles, there does exist a pebble which can be moved d vertices. We will show that this is indeed the case in Sections 3 and 4.

We note that the same techniques that are used in this article can be used to prove an interesting result about solutions of homogeneous additive equations over the field $\mathbb{Q}_2$ of 2-adic numbers. If a positive integer n is given, then the techniques in this article can be used to find the smallest number s of variables which guarantees that the equation

$$a_1 x_1^n + a_2 x_2^n + \cdots + a_s x_s^n = 0$$

always has a nontrivial 2-adic solution regardless of the (2-adic integer) values of the coefficients. In fact, the bound itself is entirely analogous to the one in our theorem here. The interested reader may refer to the article [4], which can be thought of as a companion paper to this one.

2 Preliminary Lemmata

In this section, we give the preliminary lemmata needed to prove our formula for the value of $P_n(d)$. One of our key tools is the following combinatorial lemma due to Davenport & Lewis [1].

Lemma 2. *Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, and put $a_{j+n} = a_j$ for all j. Let*

$$a_0 + a_1 + \cdots + a_{n-1} = s.$$

Then there exists a number r such that

$$a_r + \cdots + a_{r+t-1} \geq ts/n \qquad \text{for} \qquad t = 1, \ldots, n.$$

For our purposes, we can interpret this result as follows. Given an initial configuration of s pebbles on the graph, we wish to select a vertex to call v_0 and then label the vertices consecutively as $v_0, v_1, \ldots, v_{n-1}$. Let m_i represent the number of pebbles at the vertex v_i. Then Lemma 2 says that we may select v_0 so that we have $m_0 + \cdots + m_{t-1} \geq ts/n$ for $t = 1, \ldots, n$.

We now prove a lemma about the greatest integer function, which generalizes [2, Lemma 4.14]. In the proof of Theorem 1, we only need the special case where $b_i = 2$ for each i. But it is just as easy to prove the lemma in full generality, and so we do so here.

Lemma 3. *Suppose that $a_1, a_2, \ldots$ are nonnegative integers and that $b_1, b_2, \ldots$ are positive integers. Define the numbers $g_i = g_i(\mathbf{a}, \mathbf{b})$ recursively by*

$$g_1 = \left[\frac{a_1}{b_1}\right]$$

and

$$g_{i+1} = \left[\frac{g_i + a_{i+1}}{b_{i+1}}\right],$$

where $[\cdot]$ represents the greatest integer function (i.e., $[x]$ returns the greatest integer less than or equal to x). Then for all i, we have

$$g_i = \left[\frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i}\right].$$

For example, when $i = 2$ this lemma says that

$$\left\lfloor \frac{\left\lfloor \frac{a_1}{b_1} \right\rfloor + a_2}{b_2} \right\rfloor = \left\lfloor \frac{a_1}{b_1 b_2} + \frac{a_2}{b_2} \right\rfloor$$

and when $i = 3$ it says that we have

$$\left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{a_1}{b_1} \right\rfloor + a_2}{b_2} \right\rfloor + a_3}{b_3} \right\rfloor = \left\lfloor \frac{a_1}{b_1 b_2 b_3} + \frac{a_2}{b_2 b_3} + \frac{a_3}{b_3} \right\rfloor .$$

Proof. The lemma is obviously true for $i = 1$. Suppose by way of induction that it is true for a specific number i. Then we have

$$g_{i+1} \leq \frac{g_i + a_{i+1}}{b_{i+1}} < g_{i+1} + 1.$$

Our inductive hypothesis then leads to

$$b_{i+1} g_{i+1} - a_{i+1} \leq \left\lfloor \frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i} \right\rfloor < b_{i+1}(g_{i+1} + 1) - a_{i+1}.$$

Since these upper and lower bounds are both integers, this implies that we have

$$b_{i+1} g_{i+1} - a_{i+1} \leq \frac{a_1}{b_1 \cdots b_i} + \frac{a_2}{b_2 \cdots b_i} + \frac{a_3}{b_3 \cdots b_i} + \cdots + \frac{a_i}{b_i} < b_{i+1}(g_{i+1} + 1) - a_{i+1},$$

which gives

$$g_{i+1} \leq \frac{a_1}{b_1 \cdots b_i b_{i+1}} + \frac{a_2}{b_2 \cdots b_i b_{i+1}} + \frac{a_3}{b_3 \cdots b_i b_{i+1}} + \cdots + \frac{a_i}{b_i b_{i+1}} + \frac{a_{i+1}}{b_{i+1}} < g_{i+1} + 1.$$

Since g_{i+1} is an integer, the last inequality immediately implies that

$$g_{i+1} = \left\lfloor \frac{a_1}{b_1 \cdots b_i b_{i+1}} + \frac{a_2}{b_2 \cdots b_i b_{i+1}} + \cdots + \frac{a_i}{b_i b_{i+1}} + \frac{a_{i+1}}{b_{i+1}} \right\rfloor .$$

This completes the proof of the lemma.

□

We now use Lemma 3 to prove a lemma and corollary which give conditions guaranteeing that we can move a pebble from a vertex v_j on the directed cycle graph to another vertex v_{j+k}.

Lemma 4. *Consider a directed cycle graph with* n *consecutive vertices* $v_0, v_1, \ldots,$ *where the subscripts are meant to be interpreted modulo* n. *Let* m_i *be the number of pebbles at vertex* i, *and fix a vertex* v_j *and a positive integer* $k < n$. *Define numbers* $g_1, g_2, \ldots$ *inductively by* $g_1 = \left[\frac{m_j}{2}\right]$ *and*

$$g_i = \left[\frac{g_{i-1} + m_{j+i-1}}{2}\right] \qquad \text{for } i \geq 2.$$

Then we can move a pebble from v_j *to* v_{j+k} *if we have* $g_i \geq 1$ *for* $1 \leq i \leq k$.

Proof. We begin with a few simple (and perhaps overly pedantic) observations. We can move a pebble from v_j to v_{j+k} if and only if we can move it first to v_{j+1}, then to v_{j+2}, and so on until it is eventually at v_{j+k}. Next, suppose that we can move a pebble from a vertex v_j to v_{j+1}. If we can subsequently move any pebble from v_{j+1} to v_{j+2}, then we can arrange for this pebble to be the one which came from v_j. Hence, we can move a pebble from v_j to v_{j+2} if and only if we can move a pebble from v_j to v_{j+1}, and then move a pebble from v_{j+1} to v_{j+2}. (In other words, we only need to determine whether moving a pebble is possible, and don't have to keep track of which pebbles are being moved.) By induction, we can move a pebble from v_j to v_{j+k} if and only if we can first move a pebble from v_j to v_{j+1}, then move a pebble from v_{j+1} to v_{j+2}, and so on, eventually being able to move a pebble from v_{j+k-1} to v_{j+k}.

Now, when we begin making pebbling moves, we have m_j pebbles stationed at v_j. Then the number of pebbles we can move to v_{j+1} is $\left[\frac{m_j}{2}\right] = g_1$, since this is the number of disjoint pairs of pebbles at v_j. So we can move a pebble from v_j to v_{j+1} if $g_1 \geq 1$. After moving as many pebbles as possible from v_j to v_{j+1}, the number of pebbles at v_{j+1} will be $g_1 + m_{j+1}$. The maximum number of pebbles we can then move from v_{j+1} to v_{j+2} will be $\left[\frac{g_1+m_{j+1}}{2}\right] = g_2$. Hence we can move a pebble from v_j to v_{j+2} if $g_1 \geq 1$ and $g_2 \geq 1$. Continuing in this manner, we see that we can move a pebble from v_j to v_{j+k} if $g_i \geq 1$ for $1 \leq i \leq k$. This completes the proof of the lemma. □

Corollary 5. *With all variables as in Lemma 4, we can move a pebble from v_j to v_{j+k} if we have*

$$\begin{aligned} m_j &\geq 2 \\ m_j + 2m_{j+1} &\geq 4 \\ m_j + 2m_{j+1} + 4m_{j+2} &\geq 8 \\ \vdots \quad & \quad \vdots \\ m_j + 2m_{j+1} + 4m_{j+2} + \cdots + 2^{k-1}m_{j+k-1} &\geq 2^k. \end{aligned}$$

Proof. We will show that the condition $g_i \geq 1$ is equivalent to the i-th inequality in the system. We have

$$\begin{aligned} g_i &= \left\lfloor \frac{g_{i-1} + m_{j+i-1}}{2} \right\rfloor \\ &= \left\lfloor \frac{\left\lfloor \frac{g_{i-2}+m_{j+i-2}}{2} \right\rfloor + m_{j+i-1}}{2} \right\rfloor \\ &\vdots \\ &= \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{m_j}{2} \right\rfloor + \cdots}{2} \right\rfloor + m_{j+i-1}}{2} \right\rfloor \\ &= \left\lfloor \frac{m_j}{2^i} + \frac{m_{j+1}}{2^{i-1}} + \frac{m_{j+2}}{2^{i-2}} + \cdots + \frac{m_{j+i-1}}{2} \right\rfloor, \end{aligned}$$

where the last equality is true by Lemma 3.

Hence we have $g_i \geq 1$ if and only if we have

$$\frac{m_j}{2^i} + \frac{m_{j+1}}{2^{i-1}} + \frac{m_{j+2}}{2^{i-2}} + \cdots + \frac{m_{j+i-1}}{2} \geq 1.$$

This is equivalent to having

$$m_j + 2m_{j+1} + 4m_{j+2} + \cdots + 2^{i-1}m_{j+i-1} \geq 2^i,$$

which is indeed the i-th inequality. This completes the proof of the corollary. □

We finish this section with two straightforward lemmata. These are also[1] Lemmata 3.1 and 3.2 of [4]. We repeat the proofs here for completeness.

Lemma 6. *Suppose that n, N, and a are positive integers such that $aN/n > 2^a - 1$. Then we have $kN/n > 2^k - 1$ for all k satisfying $0 < k < a$.*

Proof. It is sufficient to show that $g(x) = \frac{2^x-1}{x}$ is an increasing function for $x > 0$. We have $g'(x) = \frac{x2^x \ln(2) - (2^x-1)}{x^2}$. The Mean Value Theorem implies that there exists c with $0 < c < x$ such that $\frac{2^x-1}{x} = 2^c \ln(2) < 2^x \ln(2)$. Thus $x2^x \ln(2) - (2^x - 1) > 0$ and it follows that $g'(x) > 0$.

□

Lemma 7. *Suppose that d, q are positive integers with $d \geq 2$, that r is an integer with $0 \leq r < d$, and set $n = dq + r$. Moreover, suppose that $m_0, \ldots, m_{d-2}$ are integers such that*

$$m_0 + \cdots + m_{k-1} \geq \frac{k\left((2^d-1)q + 2^r\right)}{n}$$

for $1 \leq k \leq d-1$. Then we have

$$m_0 + \cdots + m_{k-1} > 2^k - 1$$

for $1 \leq k \leq d-1$.

Proof. By Lemma 6 with $N = (2^d-1)q+2^r$, it suffices to prove the conclusion for $k = d-1$. Therefore we only need to show that

$$\frac{(d-1)\left((2^d-1)q + 2^r\right)}{n} > 2^{d-1} - 1.$$

Some algebra shows that this is true if and only if we have

$$2^{d-1}(dq - 2q - r) + (2^r(d-1) + q + r) > 0. \tag{1}$$

To see that (1) holds, we simply note that

$$\begin{aligned} &2^{d-1}(dq - 2q - r) + 2^r(d-1) + q + r \\ &\quad \geq 2^r(dq - 2q - (d-1)) + 2^r(d-1) + q + r \\ &\quad = 2^r(d-2)q + q + r \\ &\quad > 0. \end{aligned}$$

This completes the proof of the lemma.

□

[1]The statements and proofs of these lemmata were suggested by the anonymous referee of [4]. This both strengthened the lemmata from previous drafts and simplified their proofs.

3 Proof of the Theorem - The Easy Cases

In this section, we prove Theorem 1 when $d \le 2$, and also when n is divisible by d. The proof of the remaining cases is somewhat more complex and will be given in the next section.

Lemma 8. *We have $P_n(1) = n + 1$ for any n, as in Theorem 1.*

Proof. This case of Theorem 1 is trivial. As shown in the remarks after the statement of the theorem, we only need to show that $P_n(1) \le n + 1$. If there are $n + 1$ pebbles on a graph with n vertices, then some vertex has two pebbles, and a pebbling move can be made, allowing a pebble to move a distance of 1 from its starting position. Hence $P_n(1) \le n + 1$, and the proof is complete.

□

Lemma 9. *If $d|n$, with $n = dq$, then we have $P_n(d) = (2^d - 1)q + 1$, as in Theorem 1.*

Proof. As above, it suffices to show that $(2^d - 1)q + 1$ is an upper bound for $P_n(d)$. Suppose that there are exactly $(2^d - 1)q + 1$ pebbles on the graph. As discussed in the remarks after Lemma 2, we can label the vertices consecutively as $v_0, v_1, \ldots, v_{n-1}$ in such a way that if m_i represents the number of pebbles at the vertex v_i, then we have

$$m_0 + m_1 + \cdots + m_{k-1} \ge \frac{k\left((2^d - 1)q + 1\right)}{n} \tag{2}$$

for $k = 1, 2, \ldots, n$. We wish to show that it is always possible to move a pebble from v_0 to v_d. Thus, with $j = 0$ in Corollary 5, we must show that

$$m_0 + 2m_1 + \cdots + 2^{k-1}m_{k-1} \ge 2^k$$

for $1 \le k \le d$. We'll do even a little bit better than this, showing that

$$m_0 + m_1 + \cdots + m_{k-1} \ge 2^k$$

for these k. By (2), and noting that $m_0 + \cdots + m_{k-1}$ must be an integer, it suffices to show that

$$\frac{k\left((2^d - 1)q + 1\right)}{n} > 2^k - 1$$

for each k in question.

By Lemma 6, we only need to prove the inequality when $k = d$. However, this is trivial since when $k = d$ we have

$$\frac{k((2^d - 1)q + 1)}{n} = 2^d - 1 + \frac{1}{q} > 2^d - 1.$$

This completes the proof of the lemma.

□

Lemma 10. *Suppose that $d = 2$ and write $n = 2q + r$ with $0 \le r \le 1$. Then we have $P_n(2) = 3q + 2^r$, as in Theorem 1.*

Proof. If $r = 0$, then we are done by Lemma 9. Assume then that $r = 1$ and that there are exactly $3q + 2$ pebbles on the graph. Using Lemma 2, label the vertices in the same way as in the proof of Lemma 9. We will show that it is always possible to move a pebble from v_0 to v_2. It is enough to show that $m_0 \ge 2$ and $m_0 + m_1 \ge 4$. By Lemma 2, we have

$$m_0 \ge \frac{3q + 2}{2q + 1} > 1.$$

Since m_0 is an integer, this implies that $m_0 \ge 2$, as desired. Similarly, Lemma 2 yields

$$m_0 + m_1 \ge \frac{2(3q + 2)}{2q + 1} = \frac{6q + 4}{2q + 1} > 3,$$

and again we obtain the desired conclusion since $m_0 + m_1$ is an integer.

This shows that $P_n(2) \le 3q + 2^r$. As above, since we know that $P_n(2) \ge 3q + 2^r$, we must have equality, completing the proof of the lemma.

□

4 Completion of the Proof of the Theorem

In this section, we prove Theorem 1 in the case where $d \ge 3$ and $d \nmid n$. Suppose that $n = dq + r$ with $1 \le r \le d - 1$, and let $N = \left(2^d - 1\right) q + 2^r$. As before, we only need to prove that $P_n(d) \le N$, and so we suppose that there are exactly N pebbles on the graph. Label the vertices consecutively as $v_0, \ldots, v_{n-1}$ so that if m_i represents the number of pebbles on vertex v_i, then we have

$$m_0 + \cdots + m_{k-1} \geq \frac{kN}{n}$$

for all $1 \leq k \leq n$.

By Lemma 7, noting that $m_0+\cdots+m_{k-1}$ is an integer, we have $m_0+\cdots+m_{k-1} \geq 2^k$ for $1 \leq k \leq d-1$. If in addition we have $m_0 + \cdots + m_{d-1} \geq 2^d$, then by Corollary 5 we can move a pebble from v_0 to v_d, and we are done. Hence we may assume that $m_0 + \cdots + m_{d-1} \leq 2^d - 1$. This implies that

$$m_d + \cdots + m_{n-1} \geq N - \left(2^d - 1\right) = \left(2^d - 1\right)(q-1) + 2^r.$$

By Lemma 2 we can now relabel the vertices $v_d, \ldots, v_{n-1}$ as $w_0, \ldots, w_{n-d-1}$ such that both of the following properties hold.

1. The ordered tuple $(w_0, \ldots, w_{n-d-1})$ is a cyclic permutation of the ordered tuple $(v_d, \ldots, v_{n-1})$.
2. If we write m_i^* for the number of pebbles at the vertex w_i, then we have

   $$m_0^* + \cdots + m_{k-1}^* \geq \frac{k(N - 2^d + 1)}{n-d} = \frac{k\left(\left(2^d-1\right)(q-1) + 2^r\right)}{d(q-1)+r}$$

   for $1 \leq k \leq n-d$.

Lemma 7 again ensures (assuming that $q-1 \geq 1$) that $m_0^*+\cdots+m_{k-1}^* \geq 2^k$ for $1 \leq k \leq d-1$. Suppose that $w_0 = v_i$ for some i with $n-d+1 \leq i \leq n-1$. Then Corollary 5 shows that we can move a pebble from w_0 to v_0, and then as above we can move this pebble from v_0 to v_{d-1}. Since the distance from w_0 to v_{d-1} is at least d vertices, we are finished. Hence we can assume that $i \leq n-d$. That is, we can assume that the path from w_0 to w_{d-1} does not include any of the vertices $v_0, \ldots, v_{d-1}$, and this implies that the vertices $w_0, \ldots, w_{d-1}$ are consecutive as we move around the cycle graph. Corollary 5 now shows that we can move a pebble from w_0 to w_{d-1}. Again, if we have $m_0^* + \cdots + m_{d-1}^* \geq 2^d$, then we could move a pebble from w_0 even further, which would be a distance of at least d. Hence we may assume that $m_0^* + \cdots + m_{d-1}^* \leq 2^d - 1$.

We can now repeat the above argument with the remaining vertices to show that either some pebble can be moved a distance of d vertices, or else there must exist a third set of d consecutive vertices, disjoint to the two sets we have already found, which contains a total of at most $2^d - 1$ pebbles and such that we can move a pebble from the first vertex to the last. Continuing in this manner, we can in fact show that either some pebble can be moved a distance of at least d vertices or else there must exist q such mutually disjoint sets. This leaves us with r vertices for which we have not yet accounted, and these vertices will contain at least 2^r pebbles among them.

Again, we can relabel these final remaining vertices consecutively as $v_{x_0}, \ldots, v_{x_{r-1}}$ in such a way that if m_i^{**} denotes the number of pebbles on v_{x_i}, then we have $m_0^{**} + \cdots + m_{k-1}^{**} \geq k \cdot 2^r/r \geq 2^k$ for $1 \leq k \leq r$. Now fix k to be the smallest number so that $v_{x_{k+1}} \neq v_{x_k+1}$, i.e., so that the vertex following v_{x_k} on the graph is not one of the v_{x_i}. (If the vertices $v_{x_0}, \ldots, v_{x_{r-1}}$ are all consecutive, then we take $k = r - 1$.) Then the vertex v_{x_k+1} is the first vertex in one of the sets described in the previous paragraph. Corollary 5 shows that we can move a pebble from v_{x_0} to v_{x_k+1}, and then the way we constructed our sets in the previous paragraph shows that we can further move this pebble from v_{x_k+1} to v_{x_k+d}. Hence this pebble can be moved a total of at least d vertices. This completes the proof of the theorem.

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